Filtrations and Finite Model Property of $S5$

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What is Modal Logic?

- Modal logic is the logic of reasoning which involves the expressions “It is necessary that ...” and “It is possible that ...”.
- Modal logic is the logic of necessities and possibilities.

Modal Operators.

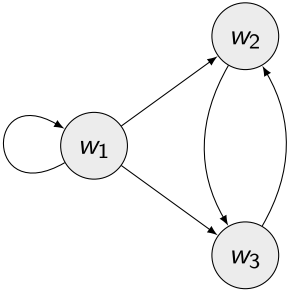
Let φ be a sentence.

- $\Box\varphi$ denotes the formula "*It is necessary that* φ".
- $\Diamond\varphi$ denotes the formula "*It is possible that* φ".

Kripke Frames.

- Consider a nonempty set W, which we'll call a set of possible worlds, or a set of states.
- Let R be a binary relation on W, which we'll call the accessibility relation.
- For any two possible worlds $\alpha, \beta \in W$, we say $\underline{\beta \text{ is accessible from } \alpha}$ iff $\alpha R \beta$.
- Pictorially, we denote $\alpha R \beta$ by drawing an arrow from α to β.

Kripke Frames - Example.



In this frame, w_1 has access to all the other possible worlds including itself, whereas w_2 has access to w_3 only.

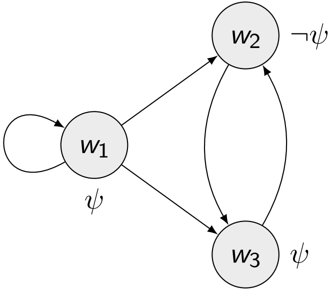
Standard Models.

- A standard model is a tuple $\mathcal{M} = (W, R, P)$, where W is a nonempty set of worlds, R is an accessibility relation on W, and $P : \mathcal{S} \to 2^W$ where $\mathcal{S}$ is the set of all propositional letters.
- The function P is an evaluation function. It assigns sets of possible worlds to atomic sentences.
- Henceforth we'll call the standard models just models.

Semantic Conditions for the modalities $\Box$ and $\Diamond$.

- Consider a model $\mathcal{M} = (W, R, P)$ and a possible world $\alpha \in W$. We have the following semantic conditions for the modalities $\Box$ and $\Diamond$. Let φ be a sentence.
- We use the notation $\mathcal{M}, \alpha \models \varphi$ to mean that φ is satisfied by the model $\mathcal{M}$ at the possible world α.
- $\mathcal{M}, \alpha \models \Box\varphi$ iff for all $\beta \in W$ with $\alpha R \beta$ we have $\mathcal{M}, \beta \models \varphi$.
- $\mathcal{M}, \alpha \models \Diamond\varphi$ iff there exists $\beta \in W$ so that $\alpha R \beta$ and $\mathcal{M}, \beta \models \varphi$.
- The semantic conditions for the logical connectives $\neg$ and $\vee$ are as usual.

Standard Models - Example.



- $\mathcal{M}, w_1 \not\models \Box\psi$ because $w_1 R w_2$ and $\mathcal{M}, w_2 \not\models \psi$.
- $\mathcal{M}, w_2 \models \Box\psi$ as the only world that can be accessed from w_2 is w_3 and we have $\mathcal{M}, w_3 \models \psi$.
- $\mathcal{M}, w_1 \models \Diamond\psi$ because R is reflexive at w_1 and ψ is true at w_1.
- $\mathcal{M}, w_3 \not\models \Diamond\psi$ as ψ is false at w_2 and w_2 is the only possible world which is accessible from w_3.

The Axiom Schema K.

- K. $\Box(p \to q) \to (\Box p \to \Box q)$.

Theorem

Suppose we have the detachment rule $p \to q, p \vdash q$ as a rule of inference. Then K *is valid.*

Proof.

Let $\mathcal{M} = (W, R, P)$ be a model and let $w \in W$. Suppose $\mathcal{M}, w \models \Box(p \to q)$. Assume $\mathcal{M}, w \models \Box p$. Let $x \in W$ be arbitrary and suppose wRx. Since $\mathcal{M}, w \models \Box(p \to q)$ and wRx, we have $\mathcal{M}, x \models p \to q$. Since $\mathcal{M}, w \models \Box p$ and wRx, we have $M, x \models p$. Hence $M, x \models q$ by detachment. Now since $x \in W$ was arbitrary with wRx, we have $\mathcal{M}, w \models \Box q$. So we conclude that K is valid in our standard models. □

Normal Modal Logic.

A set L of modal formulas such that L contains

- All propositional tautologies,
- All instances of the schema $Df\Diamond$. $\Diamond p \leftrightarrow \neg\Box\neg p$.
- All instances of the schema K. $\Box(p \to q) \to (\Box p \to \Box q)$,

and is closed under

- Detachment : $p \to q, p \vdash q$,
- Necessitation Rule : If $\vdash A$, then $\vdash \Box A$

is called a <u>Normal Modal Logic</u>.

- The smallest normal modal logic is called K.

The System $S5$.

The system $S5$ is the smallest modal logic which contains the following axiom schemas

- T. $\Box p \to p$.
- 5. $\Diamond p \to \Box\Diamond p$,

and closed under the rule $\psi_1, \ldots, \psi_n \vdash \psi$.

T and 5 are not valid in the class of all standard models.

Theorem

The schemas T. $\Box p \to p$*, and* 5. $\Diamond p \to \Box\Diamond p$ *are not valid in the class of all standard models.*

Proof.

We'll prove the invalidity of T first. Take $\mathcal{M} = (W, R, P)$, where $W = w$, $R = \emptyset$, and $P(\mathcal{S}) = \{\emptyset\}$. So in $\mathcal{M}$, there's only one possible world, it's related to nothing, and no atomic proposition is true at w. Consider the formula $\Box\mathbb{P}_0 \to \mathbb{P}_0$. Now $\mathcal{M}, w \models \Box\mathbb{P}_0$ as $R = \emptyset$. But $\mathcal{M}, w \not\models \mathbb{P}_0$ as no world is accessible from w. Hence $M, w \not\models \Box\mathbb{P}_0 \to \Diamond\mathbb{P}_0$ proving the invalidity of T.

To prove that 5 is invalid, consider the model $\mathcal{N} = (W', R', P')$, where $W' = \{w_1, w_2, w_3\}$, $R' = \{(w_1, w_2), (w_1, w_3)\}$, and $P'(\mathbb{P}_n) = \{w_2\}$ for all $n \in \mathbb{N}$. Then $\mathcal{N}, w_1 \models \Diamond\mathbb{P}_0$ while $\mathcal{N}, w_1 \not\models \Box\Diamond\mathbb{P}_0$. □

Types of R.

Consider a frame (W, R). R is said to be

- Symmetric iff uRv implies vRu for all $u, v \in W$.
- Transitive iff uRv and vRw imply uRw for all $u, v, w \in W$.
- Euclidean iff uRv and uRw imply vRw for all $u, v, w \in W$.
- An equivalence relation iff R is Reflexive, Symmetric, and Transitive.

Reflexive + Euclidean $\implies$ Equivalence Relation.

Lemma

If R is reflexive and Euclidean, then R is an equivalence relation on W.

Types of R vs T and 5.

Theorem

- *The schema* T. $\Box p \to p$ *is valid if* R *is reflexive.*
- *The schema* 5. $\Diamond p \to \Box\Diamond p$ *is valid if* R *is Euclidean.*

Validity in Frames.

- A model $\mathcal{M}$ is based on a frame $F = (W, R)$ iff $\mathcal{M} = (W, R, P)$ for some valuation P.
- For a frame F and a formula ψ, we say ψ is valid in F, denoted by $F \models \psi$, iff $\mathcal{M} \models \psi$ for all models $\mathcal{M}$ based on F.
- For a class $\mathcal{F}$ of frames, ψ is valid in $\mathcal{F}$, denoted by $\mathcal{F} \models \psi$, iff $F \models \psi$ for all $F \in \mathcal{F}$.

Frames Vs S5.

Lemma

1. *If* T. $\Box q \to q$ *is valid in a frame* $F = (W, R)$*, then* R *is relexive.*
2. *If* 5. $\Diamond q \to \Box\Diamond q$ *is valid in* $F = (W, R)$*, then* R *is Euclidean.*

Proof:

1. Suppose T is valid in $F = (R, W)$. Let $w \in W$. Our goal is to show that wRw. The idea is to build a model based on F so that we have wRw. So let $u \in P(q)$ iff wRu. Let wRu. Then $\mathcal{M}, u \models q$. So $\mathcal{M}, w \models \Box p$. Now, since $\mathcal{M}$ is a model based on F and T is valid in F, T is valid in $\mathcal{M}$ as well. So $\mathcal{M}, w \models p$. But this means wRw, done!

Proof continued.

2. For 2, suppose $F = (W, R)$ and R is not Euclidean. Then $\exists w, u, v \ (wRu \land wRv \land \neg uRv)$. We'll show that there is a model based on F so that 5 is not valid in it. Let $z \in P(q)$ iff $\neg uRz$. Now argue as in 1.

Summary of Frames Vs $S5$.

Theorem

The Reflexive + Euclidean frames characterize the system $S5$.

Corollary

The frames whose accessibility relation is an equivalence relation characterize the system $S5$.

Universal Frames.

Definition

Consider a frame $F = (W, R)$. The relation R is said to be universal iff $\forall u \forall v \ uRv$.

Some Observations :

1. Universal relations are equivalence relations.
2. An equivalence relation is a universal relation inside each equivalence class.

Logic of Universal Frames is $S5$.

Theorem

A formula ψ is valid in all frames $F = (W, R)$, where R is an equivalence relation iff ψ is valid in all frames $F = (W, R)$, where R is a universal relation.

Proof.

Since universal relations are equivalence relations the forward implication is trivial. We'll prove the reverse implication. Suppose $F = (W, R)$ is a equivalence frame and ψ is false at some $w \in W$ in a model $\mathcal{M} = (W, R, P)$ based on F. So $\mathcal{M}, w \not\models \psi$. We'll define a new model as follows.

- Let $W' = [w]$, the equivalence class of w.
- Let $R' = R \cap (W' \times W')$.
- Let $P'(q) = P(q) \cap W'$.

□

Proof Continued.

Proof.

Let $w' \in W'$. We claim $\mathcal{M}', w' \models \psi$ iff $\mathcal{M}, w' \models \psi$. We'll use induction. First suppose $\psi = q$, where q is an atom. Then $\mathcal{M}', w' \models q \iff w' \in P'(q) \iff w' \in P(q) \cap W \iff w' \in P(q)$ and $w \in W \iff \mathcal{M}, w' \models q$.

Next suppose $\psi \equiv \Box\theta$ and the result is true for θ. Then $\mathcal{M}', w' \models \Box\theta \iff \forall v' \in W'(w'R'v' \implies \mathcal{M}', v' \models \theta) \iff \forall v \in W(w'Rv \implies M, v' \models \theta) \iff \mathcal{M}, w' \models \Box\theta$. We used the IH for the second biconditional. We can prove the rest of the cases similarly.

So, $\mathcal{M}', w' \not\models \psi$, and $\mathcal{M}'$ is a model based on (W', R') with R' universal. So by contrapositive, if ψ is valid in all frames $F = (W, R)$, where R is universal, then ψ is valid in all frames $F = (W, R)$, where R is an equivalence relation.

$\square$

Filtrations.

- Fix a set of sentences Γ closed under subsentences.
- Let $\mathcal{M} = (W, R, P)$ be a model.
- Define a relation $\equiv$ on W by declaring that for any $\alpha, \beta \in W$, $\alpha \equiv \beta$ iff for every $\varphi \in \Gamma$,

$$\mathcal{M}, \alpha \models \varphi \iff \mathcal{M}, \beta \models \varphi.$$

- Then $\equiv$ is an equivalence relation on W.

Filtrations.

We say $\mathcal{M}^* = (W^*, R^*, P^*)$ is a filtration of $\mathcal{M}$ through Γ if $\mathcal{M}^*$ has the following properties.

1. $W^* = [W] = \{[\alpha] : \alpha \in W\}$.
2. For every $\alpha, \beta \in W$,
    - If $\alpha R \beta$ then $[\alpha] R^* [\beta]$,
    - If $[\alpha] R^* [\beta]$, then for every sentence $\Box\varphi \in \Gamma$, if $\mathcal{M}, \alpha \models \Box\varphi$ then $\mathcal{M}, \beta \models \varphi$,
    - If $[\alpha] R^* [\beta]$, then for every sentence $\Diamond\varphi \in \Gamma$, if $\mathcal{M}, \beta \models \varphi$ then $\mathcal{M}, \alpha \models \Diamond\varphi$.
3. $P^*(\mathbb{P}_n) = \{[\alpha] : \alpha \in P_n\}$.

Some Comments on the Definition of Filtrations.

- $P^*(\mathbb{P}_n) = \{[\alpha] : \alpha \in P_n\}$.
- If $\alpha \in P_n$, then $[\alpha] \in P^*(\mathbb{P}_n)$.
- However, $[\alpha] \in P^*(\mathbb{P}_n)$ doesn't guarantee that $\alpha \in P_n$. All we can conclude is that $\alpha \equiv \beta$ for some $\beta \in W$ with $\beta \in P_n$.
- If $\mathbb{P}_n \in \Gamma$, then we can conclude that $\alpha \in P_n$, given $[\alpha] \in P^*(\mathbb{P}_n)$.
- So whenever $\mathbb{P}_n \in \Gamma$, we'll write P_n^* for $P^*(\mathbb{P}_n)$.

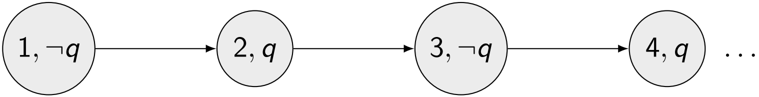
More Comments.

- If $\alpha R\beta$ then $[\alpha]R^*[\beta]$,
- If $[\alpha]R^*[\beta]$, then for every sentence $\Box\varphi \in \Gamma$, if $\mathcal{M}, \alpha \models \Box\varphi$ then $\mathcal{M}, \beta \models \varphi$,
- If $[\alpha]R^*[\beta]$, then for every sentence $\Diamond\varphi \in \Gamma$, if $\mathcal{M}, \beta \models \varphi$ then $\mathcal{M}, \alpha \models \Diamond\varphi$.
- R^* is consistent with R: R^* behaves in such a way that, for modal formulas in Γ, whose truth of course depends on the behavior of R, nothing can go wrong by any mischief that R^* may get into.

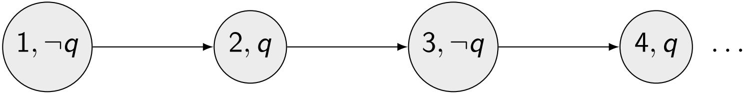
Example of a Filtration.

- Consider $\Gamma = \{q, \Box q, \Box q \to q\}$ the set of all subformulas of $\Box q \to q$, where q is an atomic formula.
- Let $W = \mathbb{Z}^+$.
- Define R by nRm iff $m = n + 1$.
- $P(q) = 2\mathbb{Z}^+$.

So we have the following infinite model $\mathcal{M}$.



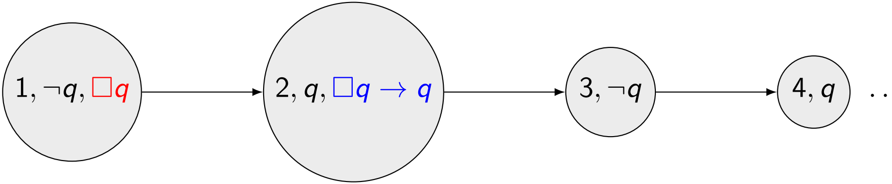
Example of a Filtration - Figuring out W^*.



Observe:

- $\mathcal{M}, n \models q$ iff n is even.
- $\mathcal{M}, n \models \Box q$ iff n is odd.
- $\therefore \mathcal{M}, n \models (\Box q \to q)$ iff n is even.
- Hence $W^* = \{[1], [2]\}$, where [1] has all and only odd numbers, and [2] has all and only even numbers.

Example of a Filtration - Figuring out P^*.



- $W^* = \{[1], [2]\}$.
- Recall $P^*(q) = \{[n] : q \in P(n)\}$.
- Since $\mathcal{M}, 2 \models q$ we have $[2] \in P^*(q)$.
- Since $\mathcal{M}, 1 \not\models q$, we have $[1] \notin P^*(q)$, because otherwise we'd have $\mathcal{M}, 1 \models q$ and hence $\mathcal{M}, 1 \models (\Box q \to q)$ which is not true.
- $\therefore\ P^*(q) = \{[2]\}$.

Example of a Filtration - Figuring out R^*.

- Since $1R2$, $[1]R^*[2]$.
- Since $2R3$ and $[3] = [1]$, $[2]R^*[1]$.
- So $\{([1],[2]),([2],[1])\} \subseteq R^*$.
- Recall: If $[\alpha]R^*[\beta]$, then for every sentence $\Box\varphi \in \Gamma$, if $\mathcal{M},\alpha \models \Box\varphi$ then $\mathcal{M},\beta \models \varphi$.
- We claim that $([1],[1]) \notin R^*$, because $\mathcal{M},1 \models \Box q$ and $[1]R^*[1]$ would force $\mathcal{M},1 \models q$, which is not true.
- Finally note that $([2],[2]) \in R^*$ or $([2],[2]) \notin R^*$ don't violate our definitions.
- So either $R^* = \{([1],[2]),([2],[2])\}$ or $R^*\{([1],[2]),([2],[1]),([2],[2])\}$.
- Thus we have two filtrations of $\mathcal{M}$ over Γ.

Example of a filtration - Summing up.

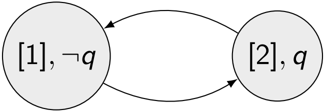


Figure 1: The filtration $\mathcal{M}_1^*$ with $R^* = \{([1],[2]),([2],[1])\}$

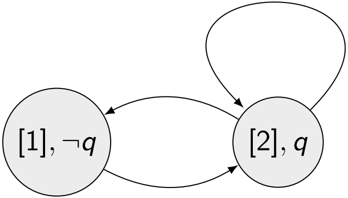


Figure 2: The filtration $\mathcal{M}_2^*$ with $R^* = \{([1],[2]),([2],[1]),([2],[2])\}$

Are Filtrations Always Finite?

- Note that we started with an infinite model and its filtrations over $\Gamma = \{q, \Box q, \Box q \to q\}$ were finite.
- And we found all possible filtrations. So we get the following question.
- So one can ask the question : Are all filtrations finite ?

Filtrations over a *Finite* Set are Finite.

Theorem

Suppose Γ is a finite set of sentences closed under subsentences. Then any filtration $\mathcal{M}^*$ of a model $\mathcal{M}$ through Γ is finite.

Proof.

We'll exhibit an injection from W^* to 2^Γ. Recall $W^* = \{[w] : w \in W\}$. Suppose $u, v \in [w]$. Then for all $\varphi \in \Gamma$, $\mathcal{M}, u \models \varphi$ iff $\mathcal{M}, v \models \varphi$. So for all $\varphi \in \Gamma$, either φ is true at both u and v, or φ is false at both u and v. Hence each $[w] \in W^*$ corresponds to some $\Lambda_w \subseteq \Gamma$. Next we claim this correspondence is injective. Assume $[x] \neq [y]$. Let $s \in [x]$, $t \in [y]$, and $\varphi \in \Gamma$ be so that $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, t \not\models \varphi$, or $\mathcal{M}, s \not\models \varphi$ and $\mathcal{M}, t \models \varphi$. But this means $\varphi \in \Gamma$, and $\varphi \in \Lambda_x$ and $\varphi \notin \Lambda_y$, or $\varphi \notin \Lambda_x$ and $\varphi \in \Lambda_y$, as claimed. Therefore $[w] \mapsto \Lambda_w$ is an injection from W^* to 2^Γ. Since Γ is finite, so is 2^Γ, and hence W^* is finite. □

Finite Model Property (FMP).

Definition

A system Σ of modal logic has the Finite Model Property if every non-theorem of Σ is false in some finite model for Σ.

M and M^* are "alike" inside Γ.

Lemma

Suppose $\mathcal{M}^*$ is a filtration of $\mathcal{M}$ through Γ. Then for all $\varphi \in \Gamma$ and $w \in W$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}^*, [w] \models \varphi$.

Proof.

Let $\varphi \in \Gamma$. We'll use an induction on the complexity of φ, and we'll look at the cases where φ is atomic and $\varphi = \Box\psi$ only, other cases are left as an exercise(?). Say $\varphi = q$, where q is an atom. Then $\mathcal{M}, w \models q \implies w \in P(q) \implies [w] \in P^*(q) \implies \mathcal{M}^*, [w] \models q$. On the other hand, $\mathcal{M}^*, [w] \models q \implies [w] \in P^*(q) \implies \exists s \in P(q)\ w \equiv s \implies \mathcal{M}, s \models q \implies \mathcal{M}, w \models q$ as $q = \varphi \in \Gamma$. $\Box$

Proof continued.

Proof.

Now suppose $\varphi = \Box\psi$ and the result is true for ψ. Assume $\mathcal{M}, w \models \Box\psi$. Further assume $[w]R^*[v]$. Then $\mathcal{M}, v \models \psi$ as $\psi \in \Gamma$ since Γ is closed under subformulas. So $\mathcal{M}^*, [v] \models \psi$. Since v was arbitrary, $\mathcal{M}^*, [w] \models \Box\psi$. Finally assume $\mathcal{M}^*, [w] \models \Box\psi$ and wRv. Then $[w]R^*[v]$. So $\mathcal{M}^*, [v] \models \psi$. Therefore $\mathcal{M}, v \models \psi$ by IH. Hence $\mathcal{M}, w \models \Box\psi$ as v was arbitrary. $\Box$

Finite Universal Models.

Lemma

Let $\mathcal{U}$ be the class of all universal models, and $\mathcal{U}_f$ be the class of all finite universal models. Then any formula ψ is valid in $\mathcal{U}$ iff ψ is valid in $\mathcal{U}_f$.

Proof.

The forward direction is trivial as $\mathcal{U}_f \subseteq \mathcal{U}$. Suppose ψ is false at a world w in some $\mathcal{M} = (W, R, P) \in \mathcal{U}$. Let

$$\Gamma = \{\phi : \phi \text{ is a subformula of } \psi\}.$$

Let $\mathcal{M}^*$ be the filtration of $\mathcal{M}$ through Γ. Since $\mathcal{M}, w \not\models \psi$ we have $\mathcal{M}^*, [w] \not\models \psi$ by the previous lemma. We'll confirm $\mathcal{M}^* \in \mathcal{U}_f$. It's finite because Γ is finite. It's universal because R is universal, and for all worlds $u, v \in W$ uRv implies $[u]R^*[v]$ by definition of R^*. □

Some Definitions.

Consider a system of modal logic Σ, and Γ a set of sentences.

Definition

- $\underline{\Gamma \text{ is } \Sigma\text{-consistent}}$:- $Con_\Sigma \Gamma$ iff $\Gamma \nvdash_\Sigma \bot$.
- $\underline{\Gamma \text{ is maximally } \Sigma\text{-consistent}}$:- $Max_\Sigma \Gamma$ iff $Con_\Sigma \Gamma$ and for all φ if $Con_\Sigma(\Gamma \cup \{\varphi\})$ then $\varphi \in \Gamma$.
- $|\varphi|_\Sigma = \{Max_\Sigma \Gamma : \varphi \in \Gamma\}$.
- $\Box\Gamma = \{\Box\varphi : \varphi \in \Gamma\}$.
- $\Diamond\Gamma = \{\Diamond\varphi : \varphi \in \Gamma\}$.
- $\Box^{-1}\Gamma = \{\varphi : \Box\varphi \in \Gamma\}$.
- $\Diamond^{-1}\Gamma = \{\varphi : \Diamond\varphi \in \Gamma\}$.

Γ vs $\Box\Gamma$.

Lemma

Let Σ be a normal system. If $\Gamma \vdash_\Sigma \varphi$ then $\Box\Gamma \vdash_\Sigma \Box\varphi$.

Proof.

Suppose $\Gamma \vdash_\Sigma \varphi$. Then there are $\psi_1, \dots, \psi_n \in \Gamma$ such that $\Sigma \vdash \psi_1 \to (\psi_2 \to \cdots (\psi_n \to \varphi) \cdots)$. Since Σ is normal, by *RK*[a] we have $\Sigma \vdash \Box\psi_1 \to (\Box\psi_2 \to \cdots (\Box\psi_n \to \Box\varphi) \cdots)$. Since $\Box\psi_i$ are in $\Box\Gamma$ we have $\Box\Gamma \vdash_\Sigma \Box\varphi$. □

[a] *RK* is the rule of inference $\dfrac{(A_1 \wedge \cdots \wedge A_n) \to A}{(\Box A_1 \wedge \cdots \wedge \Box A_n) \to \Box A}$. This can be proved using the necessitation rule and induction on n.

Lemma.

Lemma

Suppose $Max_\Sigma \Gamma$*. Then* $\Box\varphi \in \Gamma$ *iff for every* Λ *with* $Max_\Sigma \Lambda$ *and* $\Box^{-1}\Gamma \subseteq \Lambda$ *we have* $\varphi \in \Lambda$*.*

Proof.

If $\Box\varphi \in \Gamma$ and $\Box^{-1}\Gamma \subseteq \Gamma$, then $\varphi \in \Box^{-1}\Gamma \subseteq \Lambda$. For the other direction suppose $\Box\varphi \notin \Gamma$. Since $Max_\Sigma\Gamma$, $\Gamma \vdash_\Sigma \psi$ implies $\psi \in \Gamma$. So $\Gamma \nvdash_\Sigma \Box\varphi$. Therefore $\Box^{-1}\Gamma \nvdash_\Sigma \varphi$. So $Con_\Sigma(\Box^{-1}\Gamma \cup \{\neg\varphi\})$. Now if we can extend $\Box^{-1}\Gamma \cup \{\neg\varphi\}$ to a maximal Σ-consistent set then we are done. Lindenbaum's lemma takes care of this, and we'll use it without proof. So get maximal Σ-consistent Λ with $\Box^{-1}\Gamma \cup \{\neg\varphi\} \subseteq \Lambda$. Then $\varphi \notin \Lambda$ because otherwise consistency will be violated. □

Lindenbaum Lemma

For good measure we'll include the statement of the Lindenbaum Lemma.

Theorem

If $Con_\Sigma \Gamma$, then there is a Λ with $Max_\Sigma \Lambda$ extending Γ.

Canonical Models.

Let Σ be a system of modal logic.

Definition

$\mathcal{M}^{\Sigma} = (W^{\Sigma}, R^{\Sigma}, P^{\Sigma})$ is the proper canonical standard model for Σ iff

1. $W^{\Sigma} = \{\Gamma : Max_{\Sigma}\Gamma\}$,
2. For all $\alpha, \beta \in W^{\Sigma}$, $\alpha R^{\Sigma} \beta$ iff $\Box^{-1}\alpha \subseteq \beta$,
3. $P_n^{\Sigma} = |\mathbb{P}_n|_{\Sigma} = \{Max_{\Sigma}\Gamma : \mathbb{P}_n \in \Gamma\}$.

The *Truth* Lemma. I

Lemma

Let Σ *be normal. Then for all* $\Lambda \in W^{\Sigma}$, $\mathcal{M}^{\Sigma}, \Lambda \models \varphi$ *iff* $\varphi \in \Lambda$.

The *Truth* Lemma. II

Proof.

Use induction on formulas. First assume $\varphi = q$, where q is an atomic formula. Then $\mathcal{M}^\Sigma, \Lambda \models q$ iff $\Lambda \in P^\Sigma(q)$ iff $q \in \Lambda$. Next assume φ is $\neg\psi$ and the result is true for ψ. Then $\mathcal{M}^\Sigma, \Lambda \models \neg\psi$ iff $\mathcal{M}^\Sigma, \Lambda \not\models \psi$ iff $\psi \notin \Lambda$ (by IH) iff $\neg\psi \in \Lambda$ (as $Max_\Sigma\Lambda$). Now assume φ is $\mu \vee \gamma$ and the result is true for μ and γ. Then $\mathcal{M}^\Sigma, \Lambda \models \mu \vee \gamma$ iff $\mathcal{M}^\Sigma, \Lambda \models \mu$ or $\mathcal{M}^\Sigma, \Lambda \models \gamma$ iff $\mu \in \Lambda$ or $\gamma \in \Lambda$ (by IH) iff $\mu \vee \gamma \in \Lambda$ (as $Max_\Sigma\Lambda$). Finally assume φ is $\Box\psi$ and the result is true for ψ.

Suppose $\mathcal{M}^\Sigma, \Lambda \models \Box\psi$. Then for every Δ with $\Lambda R^\Sigma \Delta$ we have $\mathcal{M}^\Sigma, \Delta \models \psi$, and by IH we get $\psi \in \Delta$. But $\Lambda R^\Sigma \Delta$ means $\Box^{-1}\Lambda \subseteq \Delta$. Then by the previous lemma $\Box\psi \in \Lambda$.

Conversely suppose $\Box\psi \in \Lambda$. Let Δ be such that $\Lambda R^\Sigma \Delta$. So $\Box^{-1}\Lambda \subseteq \Delta$. So since $\Box\psi \in \Lambda$ we have $\psi \in \Box^{-1}\Lambda$. So $\psi \in \Delta$. By IH $\mathcal{M}^\Sigma, \Delta \models \psi$. Since Δ was arbitrary $\mathcal{M}^\Sigma, \Lambda \models \Box\psi$. 😄

□

Towards FMP.

Theorem

Let Σ be normal. Then $\mathcal{M}^{\Sigma} \models \psi$ iff $\Sigma \vdash \psi$.

Proof.

Suppose $\mathcal{M}^{\Sigma} \models \psi$. Then for every Λ with $Max_{\Sigma}\Lambda$, $\mathcal{M}^{\Sigma}, \Lambda \models \psi$, so by the truth lemma, $\psi \in \Lambda$. Hence $\Sigma \vdash \psi$.

Conversely suppose $\Sigma \vdash \psi$. Then for all Λ with $Max_{\Sigma}\Lambda$ we have $\psi \in \Lambda$. So again by truth lemma we have $\mathcal{M}^{\Sigma}, \Lambda \models \psi$ for every $\Lambda \in W^{\Sigma}$. Hence $\mathcal{M}^{\Sigma} \models \psi$. □

A useful lemma.

Lemma

Suppose $Max_{\Sigma}\Gamma$ *and* $Max_{\Sigma}\Lambda$*. Then* $\Box^{-1}\Gamma \subseteq \Lambda$ *iff* $\Diamond\Lambda \subseteq \Gamma$*.*

Proof.

Assume $\Box^{-1}\Gamma \subseteq \Lambda$. Suppose $\varphi \in \Lambda$. We need to show $\Diamond\varphi \in \Gamma$. By maximality of Γ, it's enough to show $\Box\neg\varphi \notin \Gamma$. If $\Box\neg\varphi \in \Gamma$, then $\neg\varphi \in \Lambda$ by assumption, and this contradicts that Λ is consistent. The other direction is proved similarly. □

The Last Step.

Lemma

Let Σ be the system $S5$. Then $\mathcal{M}^{\Sigma}$ is reflexive and Euclidean.

Proof.

Let $\Lambda_0, \Lambda_1, \Lambda_2 \in W^{\Sigma}$. First we'll show $\Lambda_0 R^{\Sigma} \Lambda_0$. Assume $\varphi \in \Box^{-1}\Lambda_0$. Then $\Box\varphi \in \Lambda_0$. So by T. $\Box\varphi \to \varphi$, $\varphi \in \Lambda_0$, and done. We're using maximality of Λ_0 to get $\Box\varphi \to \varphi \in \Lambda_0$, because if it's not in Λ_0 then we get a proper superset of Λ_0 which is Σ-consistent.

Now assume $\Lambda_0 R^{\Sigma} \Lambda_1$ and $\Lambda_0 R^{\Sigma} \Lambda_2$. We'll show $\Lambda_1 R^{\Sigma} \Lambda_2$. We have $\Box^{-1}\Lambda_0 \subseteq \Lambda_1, \Lambda_2$. Equivalently, by the previous lemma, $\Diamond\Lambda_1, \Diamond\Lambda_2 \subseteq \Lambda_0$. We claim $\Diamond\Lambda_2 \subseteq \Lambda_1$. Let $\Diamond\varphi \in \Diamond\Lambda_2$. Then $\varphi \in \Lambda_2$. So $\Diamond\varphi \in \Lambda_0$. By 5. $\Diamond\varphi \to \Box\Diamond\varphi$, we have $\Box\Diamond\varphi \in \Lambda_0$. So $\Diamond\varphi \in \Lambda_1$, and the claim is proved. Now, again, by the previous lemma, $\Box^{-1}\Lambda_1 \subseteq \Lambda_2$, or rather $\Lambda_1 R^{\Sigma} \Lambda_2$. Hence R^{Σ} is Euclidean. $\Box$

Finite Model Property (FMP).

Now we're ready to prove our main result. Recall:

Definition

A system Σ of modal logic has the Finite Model Property if every non-theorem of Σ is false in some finite model for Σ.

S5 has FMP.

Theorem

The system S5 has the finite model property.

Proof.

Denote the system S5 by Σ. Let φ be a formula. Suppose $\Sigma \nvdash \varphi$. Then $\mathcal{M}^{\Sigma} \nvDash \varphi$ as S5 is normal. So since $\mathcal{M}^{\Sigma}$ is reflexive and Euclidean, there is a universal model $\mathcal{N}$ so that $\mathcal{N} \nvDash \varphi$. Take the filtration $\mathcal{N}^*$ of $\mathcal{N}$ over the finite set

$$\Gamma = \{\psi : \psi \text{ is a subformula of } \varphi\}.$$

Then, since $\mathcal{N}^*$ and $\mathcal{N}$ are "alike" inside Γ, we have $\mathcal{N}^* \nvDash \varphi$. And since $\mathcal{N}$ is universal and Γ is finite, $\mathcal{N}^*$ is finite universal. So $\mathcal{N}^*$ is a finite model of S5 with $\mathcal{N}^* \nvDash \varphi$. Hence S5 has FMP. □

References

- Modal Logic an Introduction, B.F. Chellas
- Boxes and Diamonds, The Open Logic Project