α-theory

649B Class Project

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Introduction...

- In their paper "Alpha-Theory: An Elementary Axiomatics for Nonstandard Analysis" V. Benci and M. Di Nasso present the methods of NSA by postulating a few properties for an infinite "ideal" number α.
- We can think of this as postulating a set of axioms for the imaginary number i and building $\mathbb{C}$ on it.
- Here, I try to present what I learned from the paper and discuss how they are related to what we learned in class.

The Five Axioms...*Informally*

α1. Extension Axiom: For every sequence φ there is a unique element $\varphi[\alpha]$, called the ideal value of φ or the *value of* φ *at infinity*.

α2. Composition Axiom: For sequences φ and ψ and if f is a function such that the compositions $f \circ \varphi$ and $f \circ \psi$ make sense, then $\varphi[\alpha] = \psi[\alpha] \implies (f \circ \varphi)[\alpha] = (f \circ \psi)[\alpha]$.

α3. Number Axiom: If $c_r : n \mapsto r$ is the constant sequence with value $r \in \mathbb{R}$, then $c_r[\alpha] = r$. If $1_{\mathbb{N}} : n \mapsto n$ is the identity sequence on $\mathbb{N}$, then $1_{\mathbb{N}}[\alpha] = \alpha \notin \mathbb{N}$.

α4. Pair Axiom: For all sequences φ, ψ, and θ, if for all n $\theta(n) = \{\varphi(n), \psi(n)\}$, then $\theta[\alpha] = \{\varphi[\alpha], \psi[\alpha]\}$.

α5. Internal Set Axiom: If ψ is a sequence of atoms[1], then $\psi[\alpha]$ is an atom. If $c_\emptyset : n \mapsto \emptyset$ is the constant sequence taking the value the empty set, then $c_\emptyset[\alpha] = \emptyset$. If ψ is a sequence of nonempty sets, then $\psi[\alpha] = \{\varphi[\alpha] : \varphi(n) \in \psi(n)\ \forall n\}$.

[1] same as urelements

Some Comments on the Five Axioms...

- Axiom $\alpha 3$ gives a new number α which is not a natural number. We'll see that this is not even a real number!
- Does the Compactness Theorem justify the existence of α?
- We'll also see that the ideal value of a sequence $\varphi : \mathbb{N} \to A$ is the value of the extended sequence $\varphi : {}^*\mathbb{N} \to {}^*A$ at α.

Some consequences of the five axioms...

♣ φ and ψ are sequences of non-empty sets throughout.

1. Union: $\theta(n) = \varphi(n) \cup \psi(n) \implies \theta[\alpha] = \varphi[\alpha] \cup \psi[\alpha]$.
2. Subset: $\varphi(n) \subseteq \psi(n) \implies \varphi[\alpha] \subseteq \psi[\alpha]$.
3. Ordered Pair: $\theta(n) = (\varphi(n), \psi(n)) \implies \theta[\alpha] = (\varphi[\alpha], \psi[\alpha])$.
4. Cartesian Product: $\theta(n) = \varphi(n) \times \psi(n) \implies \theta[\alpha] = \varphi[\alpha] \times \psi[\alpha]$.
5. Difference: $\varphi(n) \neq \psi(n) \implies \varphi[\alpha] \neq \psi[\alpha]$, and $\varphi(n) \notin \psi(n) \implies \varphi[\alpha] \notin \psi[\alpha]$.
6. Setminus: $\theta(n) = \varphi(n) \setminus \psi(n) \implies \theta[\alpha] = \varphi[\alpha] \setminus \psi[\alpha]$.
7. Intersection: $\theta(n) = \varphi(n) \cap \psi(n) \implies \theta[\alpha] = \varphi[\alpha] \cap \psi[\alpha]$.

Proof.

1. Suppose $\zeta \in \varphi[\alpha] \cup \psi[\alpha]$. Then either $\zeta \in \varphi[\alpha]$ or $\zeta \in \psi[\alpha]$. Assume $\zeta \in \varphi[\alpha]$. Then, by the internal set axiom, $\zeta = \widetilde{\zeta}[\alpha]$ for some sequence $\widetilde{\zeta}$ with $\widetilde{\zeta}(n) \in \varphi(n)$ for all n. So, $\widetilde{\zeta}(n) \in \varphi(n) \cup \psi(n) = \theta(n)$ for all n. Hence, by the internal set axiom, $\zeta \in \theta[\alpha]$. Similarly, if $\zeta \in \psi[\alpha]$, then $\zeta \in \theta[\alpha]$, and therefore $\varphi[\alpha] \cup \psi[\alpha] \subseteq \theta[\alpha]$.

Now suppose $\zeta \in \theta[\alpha]$. Then $\zeta = \widetilde{\zeta}[\alpha]$, where $\widetilde{\zeta}(n) \in \theta(n) = \varphi(n) \cup \psi(n)$ for all n. Define a sequence η as follows:

$$\eta(n) = \begin{cases} \varphi(n) & \text{if } \widetilde{\zeta}(n) \in \varphi(n) \\ \psi(n) & \text{if } \widetilde{\zeta}(n) \in \psi(n) \setminus \varphi(n). \end{cases}$$

Then $\eta(n) \in \{\varphi(n), \psi(n)\}$ for all n. Put $\eta'(n) = \{\varphi(n), \psi(n)\}$ for all n. Then, by the pair axiom, $\eta'[\alpha] = \{\varphi[\alpha], \psi[\alpha]\}$. But $\eta(n) \in \eta'(n)$ for all n. So, by the internal set axiom, $\eta[\alpha] \in \eta'[\alpha]$. So, $\eta[\alpha] = \varphi[\alpha]$ or $\eta[\alpha] = \psi[\alpha]$. But $\widetilde{\zeta}(n) \in \eta(n)$ for all n. So, again by the internal set axiom, $\widetilde{\zeta}[\alpha] \in \eta[\alpha]$. Therefore, $\widetilde{\zeta}[\alpha] = \varphi[\alpha]$ or $\widetilde{\zeta}[\alpha] = \psi[\alpha]$. Hence $\zeta = \widetilde{\zeta}[\alpha] \in \varphi[\alpha] \cup \psi[\alpha]$, and done!

For the proofs of the other results see the Appendix B.

Ideal Values behave nicely...

1. If φ is a sequence of atoms (or sets, or nonempty sets) then $\varphi[\alpha]$ is an atom (or a set, a nonempty set, respectively).

2. If the value at infinity $\varphi[\alpha]$ is an atom (or a set, or a nonempty set) then there exists a sequence $\psi(n)$ of atoms (of sets, of nonempty sets, respectively) so that $\psi[\alpha] = \varphi[\alpha]$.

3. Elements of values at infinity are values at infinity.

Ideal Values continue to behave nicely...

1. If the sequence φ and ψ agree eventually, then $\varphi[\alpha] = \psi[\alpha]$.
2. If the sequences φ and ψ differ eventually, then $\varphi[\alpha] \neq \psi[\alpha]$.

<u>Comment</u>: By the property 2, the ideal values of the two sequences $\langle 1/n \rangle$ and $\langle 1/(n+1) \rangle$ must be different! But this was a little counter-intuitive to me at first because these two sequences have the same value at infinity which I thought was their limit! However, the notion of "shadows" cleared my confusion. We'll see it shortly.

Proof.

1. Let $\{n : \varphi(n) \neq \psi(n)\} = \{n_1, \ldots, n_k\}$. Fix m with $\varphi(m) = \psi(m)$. Put

$$\zeta(n) = \begin{cases} n & \text{if } \varphi(n) \neq \psi(n) \\ m & \text{if } \varphi(n) = \psi(n). \end{cases}$$

Then by the pair axiom $\zeta[\alpha] = \alpha$ or $\zeta[\alpha] = m$. But $\zeta(n) \in \{n_1, \ldots, n_k, m\}$ and $\alpha \notin \mathbb{N}$, whence $\zeta[\alpha] = m$. Let A be the range of φ. Put

$$\eta(n) = \begin{cases} A & \text{if } n \neq m \\ \emptyset & \text{if } n = m. \end{cases}$$

Now, $(\eta \circ \zeta)(n) = A$, and therefore $\{\varphi(n)\} \setminus \{\psi(n)\} \subseteq A$ as A is the range of φ. Therefore $\{\varphi[\alpha]\} \setminus \{\psi[\alpha]\} \subseteq (\eta \circ \zeta)[\alpha]$. But $\zeta[\alpha] = m = c_m[\alpha]$. So, $(\eta \circ \zeta)[\alpha] = (\eta \circ c_m)[\alpha] = c_\emptyset[\alpha] = \emptyset$ by the composition axiom. Hence $\{\varphi[\alpha]\} \setminus \{\psi[\alpha]\} = \emptyset$ or rather $\varphi[\alpha] = \psi[\alpha]$. $\square$

Proof.

2. $\varphi(n) \neq \psi(n)$ for all but finitely many n. So, $\{\varphi(n)\} \setminus \{\psi(n)\} = \{\varphi(n)\}$ for all but finitely many n. Therefore $\{\varphi[\alpha]\} \setminus \{\psi[\alpha]\} = \{\varphi[\alpha]\}$. Hence $\varphi[\alpha] \neq \psi[\alpha]$. $\square$

Resemblances... Agreeing on a "big set" is enough?

Suppose $\varphi[\alpha] = \psi[\alpha]$ and let $\Lambda = \{n : \varphi(n) = \psi(n)\}$. Then for all sequences ζ and η we have

1. If $\zeta(n) = \eta(n)$ for all $n \in \Lambda$, then $\zeta[\alpha] = \eta[\alpha]$.
2. If $\zeta(n) \in \eta(n)$ for all $n \in \Lambda$, then $\zeta[\alpha] \in \eta[\alpha]$.

The corresponding results hold when $=$ and $\in$ are replaced with $\neq$ and $\notin$ respectively in Λ, and 1 and 2. Obviously, the $\in$ in the quantifiers remain unchanged.

Comment: (We'll use this idea later in our proof of countable saturation) If $\varphi[\alpha] \neq \psi[\alpha]$, we can assume without loss of generality that $\varphi(n) \neq \psi(n)$ for all n. Here's why. Put $\Lambda = \{n : \varphi(n) \neq \psi(n)\}$. Define $\varphi'(n) = \varphi(n)$ if $n \in \Lambda$ and $\varphi'(n) \neq \psi(n)$ otherwise. Then $\varphi'(n) \neq \psi(n)$ for all n. Also, $\varphi'[\alpha] = \varphi[\alpha]$.

Proof.

We'll look at the proof of 1 only. For the proofs of the other properties see the Appendix B. Suppose $\zeta(n) = \eta(n)$ for all $n \in \Lambda$. Define

$$\theta(n) = \begin{cases} \{\zeta(n)\} = \{\eta(n)\} & \text{if } n \in \Lambda \\ \emptyset & \text{otherwise .} \end{cases}$$

Observe that $(\{\varphi(n)\} \setminus \{\psi(n)\}) \cup \theta(n) = \{\varphi(n)\}$ outside Λ, and $(\{\varphi(n)\} \setminus \{\psi(n)\}) \cup \theta(n) = \{\zeta(n)\}$ inside Λ. So, $(\{\varphi(n)\} \setminus \{\psi(n)\}) \cup \theta(n) \neq \emptyset$. Therefore, since $\varphi[\alpha] = \psi[\alpha]$, we have $(\{\varphi[\alpha]\} \setminus \{\psi[\alpha]\}) \cup \theta[\alpha] = \theta[\alpha] \neq \emptyset$. By the pair axiom $\theta[\alpha] = \emptyset$ or $\theta[\alpha] = \{\zeta[\alpha]\} = \{\eta[\alpha]\}$. But since $\theta[\alpha] \neq \emptyset$ we have $\theta[\alpha] = \{\zeta[\alpha]\} = \{\eta[\alpha]\}$, or rather $\zeta[\alpha] = \eta[\alpha]$. $\square$

The Star-Operator

- Introduces a notion of "idealization" for any entity A, namely the *nonstandard extension* or *star-transform* ${}^*A$.[2]

Definition

For any entity A, ${}^*A = c_A[\alpha]$, the value at infinity taken by the constant sequence $c_A : n \mapsto A$.

Comments: By the Number Axiom ${}^*r = r$ for all reals r.

By the internal set axiom $c_A[\alpha] = \{\psi[\alpha] : \psi(n) \in c_A(n)\ \forall n\} = \{\psi[\alpha] : \psi(n) \in A\ \forall n\} = \{\psi[\alpha] : \psi : \mathbb{N} \to A\}$. Hence

$$ {}^*A = \{\psi[\alpha] : \psi : \mathbb{N} \to A\}. $$

[2] In the paper the authors use A^*, but I will use ${}^*A$ as that's what we used in class.

Star-operator preserves basic operations of sets except the powerset...

To see $A = B \implies {}^*A = {}^*B$, observe that $A = B \implies c_A(n) = c_B(n)$ for all $n \iff {}^*A = c_A[\alpha] = c_B[\alpha] = {}^*B$. Similarly other set operations are preserved by the star-operator. In sum we have:

- $A = B \iff {}^*A = {}^*B$
- $A \in B \iff {}^*A \in {}^*B$
- $A \subseteq B \iff {}^*A \subseteq {}^*B$
- ${}^*\{A, B\} = \{{}^*A, {}^*B\}$
- ${}^*(A, B) = ({}^*A, {}^*B)$
- ${}^*(A \cup B) = {}^*A \cup {}^*B$
- ${}^*(A \cap B) = {}^*A \cap {}^*B$
- ${}^*(A \setminus B) = {}^*A \setminus {}^*B$
- ${}^*(A \times B) = {}^*A \times {}^*B$

Star-transform of a function...

For a function $f : A \to B$, its star-transform ${}^*f : {}^*A \to {}^*B$ is a function so that for every sequence $\varphi : \mathbb{N} \to A$ we have ${}^*f(\varphi[\alpha]) = (f \circ \varphi)[\alpha]$.

Comment: The ideal value of a sequence $\varphi : \mathbb{N} \to A$ is the value of the extended sequence $\varphi : {}^*\mathbb{N} \to {}^*A$ at α, i.e.

$$\varphi[\alpha] = (\varphi \circ 1_{\mathbb{N}})[\alpha] = {}^*\varphi(1_{\mathbb{N}}[\alpha]) = {}^*\varphi[\alpha].$$

The Hyperreal Line...

Definition

The set of hyperreal numbers is the star transform ${}^*\mathbb{R}$ of the set of real numbers, i.e.

$$ {}^*\mathbb{R} = \{\varphi[\alpha] : \varphi : \mathbb{N} \to \mathbb{R}\}. $$

Comment: By the number axiom every real number is a hyperreal number. So $\mathbb{R} \subseteq {}^*\mathbb{R}$. In fact, this inclusion is proper! Proof, shortly.

For more details see sections 2.2, 2.3, and 2.4. which I will skip as the aim of this presentation is to discuss more about the foundational aspects of the theory developed by the authors.

The Hypernatural Numbers...

Definition

The set of hypernatural numbers is the star-transform of the set of natural numbers, i.e.

$$^*\mathbb{N} = \{\varphi[\alpha] : \varphi : \mathbb{N} \to \mathbb{N}\}.$$

Clearly $^*\mathbb{N} \subseteq {}^*\mathbb{R}$. Now let's see why $\mathbb{R} \subsetneq {}^*\mathbb{R}$. For every $k \in \mathbb{N}$, $\alpha = 1_{\mathbb{N}}(\alpha) > c_k(\alpha) = k$. So $\alpha \in {}^*\mathbb{R} \setminus \mathbb{R}$.

Some nice facts about hypernaturals...

1. The natural numbers are a proper initial segment of the hypernatural numbers i.e. $\mathbb{N} \subsetneq {}^*\mathbb{N}$ and for every $\zeta \in {}^*\mathbb{N}$, $\zeta < n \in \mathbb{N} \implies \zeta \in \mathbb{N}$.

2. The hypernatural numbers are unbounded in the hyperreal line.

3. For every $\zeta \in {}^*\mathbb{N}$, there are no hypernatural numbers η strictly between ζ and $\zeta + 1$.

For more details see sections 2.2, 2.3, and 2.4. which I will skip as the aim of this presentation is to discuss more about the foundational aspects of the theory developed by the authors. Section 3 introduces infinitesimals, infinitely large numbers, and nonstandard calculus.

Shadow Theorem...

Theorem

Every finite[a] *hyperreal number* ζ *is infinitely close to a unique real number* r*, called the shadow of* ζ*. Symbolically,* $r = sh(\zeta)$*.*[b]

[a] finite hyperreals are defined in the same way we defined them in class.
[b] Shadow of ζ is the same as the standard part of ζ.

Proof.

This is Theorem 3.5. □

Confusions revisited...

We had the following comment.

Comment: By the property 2, the ideal values of the two sequences $\langle 1/n \rangle$ and $\langle 1/(n+1) \rangle$ must be different! But this was a little counter-intuitive to me at first because these two sequences have the same value at infinity which I thought was their limit! However, the notion of "shadows" cleared my confusion. We'll see it shortly.

The simple answer is the ideal values (or the values at infinity) and limits are not the same thing.

We have if $\lim_n \varphi(n) = \ell$ then $sh(\varphi[\alpha]) = \ell$. However, if $sh(\varphi[\alpha]) = \ell$ then all we can say is that there is a *subsequence* of $\varphi(n)$ with limit ℓ.

Some Calculus...

One difference with the Alpha-Theory approach to calculus is that it doesn't appeal to transfer. Let's look at an example.

$f : A \to \mathbb{R}$ is a function and A contains a nbhd of x_0. Then f is continuous at x_0 if for every $\zeta \in {}^*A$, $\zeta \sim x_0 \implies {}^*f(\zeta) \sim f(x_0)$.

We'll prove the intermediate value theorem. $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < 0$ and $f(b) > 0$. Need to show that there is $x_0 \in (a, b)$ with $f(x_0) = 0$.

For each $n > 0$ put $A(n) = \{a + i\frac{b-a}{n} : i = 0, \ldots, n-1\}$. $A(n)$ partitions $[a, b]$ into n intervals of equal length. Put $\eta(n) = \max\{x \in A(n) : f(x) < 0\}$ and $\zeta(n) = \eta(n) + \frac{b-a}{n}$. Then $\zeta(n) \in [a, b]$ (worst case is $\eta(n) = (n-1)\frac{b-a}{n}$). And $f(\zeta(n)) \geq 0$ because $\zeta(n) > \eta(n)$.

$\zeta(n) - \eta(n) = b - a/n$ for all n. So $\zeta[\alpha] - \eta[\alpha] = \frac{b-a}{\alpha} \sim 0$. Therefore $\zeta[\alpha] \sim \eta[\alpha]$. So $Sh(\zeta[\alpha]) = Sh(\eta[\alpha]) = x_0$ for some $x_0 \in [a, b]$.

Some Calculus...ctd

${}^*f(\zeta[\alpha]) \sim f(x_0) \sim {}^*f(\eta[\alpha])$ by continuity. Since $f(\zeta(n)) \geq 0$ for all n we have ${}^*f(\zeta[\alpha]) \geq 0$, and since $f(\eta(n)) < 0$ for all n we have ${}^*f(\eta[\alpha]) < 0$. This forces $f(x_0) = 0$.

Internal Sets...

Definition

An entity is internal if it is the ideal value of some sequence. An entity is external if it is not internal.

- Every nonstandard extension is internal.
- α is internal by the number axiom.

Comment: Recall that, in class, we defined A to be an internal set if $A \in {}^*B$ for some $B \in V(S)$. But we saw that ${}^*B = \{\varphi[\alpha] : \varphi \in {}^{\mathbb{N}}B\}$. This is so by the *internal set axiom*. Hence if $A \in {}^*B$ then $A = \varphi[\alpha]$ for some sequence φ. This kinda justifies why "the internal set axiom" is called the internal set axiom.

Countable Saturation...

Theorem

If $\{A_k : k \in \mathbb{N}\}$ is a countable family of internal sets with FIP, then $\bigcap_{k\in\mathbb{N}} A_k \neq \emptyset$.

Proof: Let $k \in \mathbb{N}$ be arbitrary. Then $A_k = \varphi_k[\alpha]$ for some sequence $\langle \varphi_k(n) \rangle_{n\in\mathbb{N}}$. By hypothesis, $\varphi_1[\alpha] \cap \cdots \cap \varphi_k[\alpha] = A_1 \cap \cdots \cap A_k \neq \emptyset = 1_\emptyset[\alpha]$. So, we may assume without loss of generality that $\varphi_1(n) \cap \cdots \cap \varphi_k(n) \neq 1_\emptyset(n)$ for all n. Define the sequence $\langle \psi(n) \rangle$ by letting $\psi(n) \in \varphi_1(n) \cap \cdots \cap \varphi_k(n)$ for all n. Then $\psi[\alpha] \in \varphi_1[\alpha] \cap \cdots \varphi_k[\alpha]$. Since k was arbitrary, the proof is complete. □

Overflow and Underflow...

- These principles are proved using saturation. See Proposition 4.6.

The Foundations

We work in the language $\mathcal{L} = \{\in, \mathcal{A}, J\}$, and write the axioms of the Alpha-Theory. This is the language of set theory with a set $\mathcal{A}$ of atoms ($\mathcal{A}$ is a constant symbol), and a binary relation symbol J.

J1. *Extension Axiom*. If φ is a sequence, then there is a unique x so that $J(\varphi, x)$. If $J(\varphi, x)$ for some x, then φ is a sentence.[3]

J2. *Composition Axiom*. If φ, ψ are sequences and if f is any function such that $f \circ \varphi$ and $f \circ \psi$ make sense, then

$$\forall x[(J(\varphi, x) \wedge J(\psi, x)) \rightarrow \exists y(J(f \circ \varphi, y) \wedge J(f \circ \psi, y))].$$

J3. *Number Axiom*. Let $r \in \mathbb{R} \subseteq \mathcal{A}$. If $c_r : n \mapsto r$ is the sequence with value r, then $\forall x[J(c_r, x) \rightarrow x = r]$. If $1_{\mathbb{N}} : n \mapsto n$ is the identity sequence on $\mathbb{N}$, then $\forall x[J(1_{\mathbb{N}}, x) \rightarrow x \notin \mathbb{N}]$.

[3] This tells us that J is a function defined on the class of all sequences.

J4. *Pair Axiom.* For all sequences φ, ψ, θ with $\theta(n) = \{\varphi(n), \psi(n)\}$ for all n we have

$$\forall x \forall y \forall z [(J(\varphi, x) \wedge J(\psi, y) \wedge J(\theta, z)) \rightarrow z = \{x, y\}].$$

J5. *Internal Set Axiom.*[4] If ψ is a sequence of atoms, then $\forall x[J(\varphi, x) \rightarrow x \in \mathcal{A}]$. If $c_\emptyset$ is the constant sequence with value the empty set, then $J(c_\emptyset, \emptyset)$. If φ is a sequence of nonempty sets, then

$$\forall x[J(\varphi, x) \rightarrow \forall y(y \in x \leftrightarrow \exists \psi(\psi E \varphi \wedge J(\psi, y)))].$$

[4] $\psi E \varphi$ is an abbreviation for $\psi(n) \in \varphi(n)$ for all $n \in \mathbb{N}$.

The Foundations

Definition

The Alpha-Theory is the first order theory in the language $\mathcal{L} = \{\in, \mathcal{A}, J\}$, whose axioms consist of:

- All axioms of ZFCA[a], with the only exception of the foundation axiom. Separation and replacement schemes are also considered for formulas with the symbol J.
- The five axioms **J1,J2,J3,J4,J5**.

[a] Zermelo-Fraenkel set theory with atoms

An important ultrafilter...

Lemma

Put $U_\alpha = \{A \subseteq \mathbb{N} : \alpha \in {}^*A\}$. Then U_α is a nonprincipal ultrafilter.

Proof.

Obviously $\mathbb{N} \in U_\alpha$, and $\emptyset \notin U_\alpha$ as ${}^*\emptyset = \emptyset$. Since $A \subseteq B \implies {}^*A \subseteq {}^*B$, U_α "filters sets upwards". And since ${}^*(A \cap B) = {}^*A \cap {}^*B$, U_α is closed under finite intersections. Hence U_α is a filter. Suppose $A \subseteq \mathbb{N}$ and $A \notin U_\alpha$. Then $\alpha \notin {}^*A$. So, $\alpha \in {}^*\mathbb{N} \setminus {}^*A = {}^*(\mathbb{N} \setminus A)$. Since $\mathbb{N} \setminus A \subseteq \mathbb{N}$, $\mathbb{N} \setminus A \in U_\alpha$. Hence U_α is an ultrafilter. Finally, if U_α were principal, then we'd have some $n_0 \in \mathbb{N}$ so that $U_\alpha = \{A \subseteq \mathbb{N} : n_0 \in A\}$, and in particular $\{n_0\}$ would be in U_α, which is impossible as $\alpha \neq n$ for any $n \in \mathbb{N}$ and ${}^*\{n_0\} = \{n_0\}$. This completes the proof. $\square$

Alpha-Theory proves the transfer principle...

Theorem

For every bounded quantifier formula $\sigma(x_1, \ldots, x_k)$ in the language of set theory, and for every $a_1, \ldots, a_k$,

$$\sigma(a_1, \ldots, a_k) \iff \sigma({}^*a_1, \ldots, {}^*a_k).$$

Proof idea: They prove that if φ_i are sequences and $\sigma(x_1, \ldots, x_k)$ is a bounded quantifier formula, then

$$\sigma(\varphi_1[\alpha], \ldots, \varphi_k[\alpha]) \iff \alpha \in {}^*\{n : \sigma(\varphi_1(n), \ldots, \varphi_k(n))\}.$$

[5] But ${}^*a_j = c_{a_j}[\alpha]$, where c_{a_j} is the constant sequence with the value a_j. Then the following observation is made. $\{n : \sigma(c_{a_1}(n), \ldots, c_{a_k}(n))\}$ is empty if $\sigma(a_1, \ldots, a_n)$ fails, and it's $\mathbb{N}$ if $\sigma(a_1, \ldots, a_k)$ holds. So, $\sigma(a_1, \ldots, a_n) \iff \alpha \in {}^*\mathbb{N} = {}^*\{n : \sigma(c_{a_1}(n), \ldots, c_{a_k}(n))\} \iff$ $\sigma({}^*a_1, \ldots, {}^*a_k)$, as $c_{a_i}[\alpha] = {}^*a_i$ for each i.

[5] Compare with the ultrafilter U_α.

Zermelo-Fraenkel-Boffa set theory with Choice **ZFBC**...What is it???

Definition

ZFBC is the theory $\mathbf{ZFC}^- + \mathbf{GAC} + \mathbf{BA}$.

ZFC^- is Zermelo-Fraenkel set theory without the axiom of foundation.

Introduce a new binary relation symbol C to the language of ZFC^- so that the axiom schemes of separation and collection apply to the formulas in the extended language. Then GAC defines a one-to-one correspondence between ordinals and sets. Namely,

$(\forall x)[x \text{ is an ordinal number } \implies (\exists! y) C(x,y)] \wedge (\forall y)(\exists! x) C(x,y) \wedge (\forall x)(\forall y)[C(x,y) \implies x \text{ is an ordinal number}]$.

ZFBC ctd...

BA, aka the *axiom of superuniversality* is the following.

If (A, R) is transitive in an extensional (A', R'), B is transitive, and $f : (A, R) \to (B, \in_B)$ is an isomorphism, then there exist B' and f' so that $B \subseteq B',\ f \subseteq f',\ B'$ is transitive and $f' : (A', R') \to (B', \in_{B'})$ is an isomorphism.

For more details see "Standard Foundations for Nonstandard Analysis" by D. Ballard and K. Hrbacek.

Models of the Alpha-Theory

Theorem

1. ZFBC *proves the following: For each nonprincipal ultrafilter* D *on* $\mathbb{N}$, *a function* J_D *can be defined on the class of all sequence in such a way that the internal model* $\mathcal{M}_D = (V, \in, J_D)$ *is a model of the Alpha-Theory and* $\mathcal{M}_D \models$ "$U_\alpha = D$".
2. *Let* $\mathcal{U}$ *be a countable model of* ZFC *and assume that*

$$\mathcal{U} \models \text{"}D \text{ is a nonprincipal ultrafilter on } \mathbb{N}\text{"}.$$

Then $\mathcal{U}$ *is the well-founded part of some model* $\mathcal{N}_D$[a] *of the Alpha-Theory such that* $\mathcal{N}_D \models$ "$U_\alpha = D$".

[a] That is $\mathcal{U}$ is the submodel of $\mathcal{N}_D$ whose universe is $\{x \in \mathcal{N}_D : \mathcal{N}_D \models \text{"}x \text{ is wellfounded"}\}$

We will not discuss the proof of this theorem, but use it to discuss the proofs of some other nice results.

α-theory proves a theorem in ordinary maths iff it is true...

Consider a sentence σ in the language of set theory. Then the sentence σ^{WF} is formed as follows.

Every quantifier $\exists x \ldots$ occurring in σ is replaced by $\exists x(x \text{ is wellfounded } \wedge \ldots)$.

Every quantifier $\forall x \ldots$ is replaced by $\forall x(x \text{ is wellfounded } \rightarrow \ldots)$

α-theory proves a theorem in ordinary maths iff it is true...

Theorem

A sentence σ in the language of set theory is a theorem of ZFC *iff σ^{WF} is a theorem of the Alpha-Theory.*

Proof.

Suppose ZFC $\nvDash \sigma$. Downward Löwenheim-Skolem says there's a countable model $\mathcal{U}$ of ZFC with $\mathcal{U} \models \neg\sigma$. Let D be a nonprincipal ultrafilter over $\mathbb{N}$ with $\mathcal{U} \models$ "D is a nonprincipal ultrafilter over $\mathbb{N}$". By part 2 of the previous theorem, $\mathcal{U}$ is the wellfounded part of some model $\mathcal{N}_D$ of the Alpha-Theory. That is $\mathcal{U}$ is the submodel of $\mathcal{N}_D$ whose universe is $\{x \in \mathcal{N}_D : \mathcal{N}_D \models$ "x is wellfounded" $\}$. So, $U \models \neg\sigma \iff \mathcal{N}_D \models (\neg\sigma)^{WF} = \neg(\sigma)^{WF}$. Hence (Alpha-Theory) $\nvDash \sigma^{WF}$.

Suppose (Alpha-Theory)$\nvDash \sigma^{WF}$. Let $\mathcal{M}$ be a model of the Alpha-Theory with $\mathcal{M} \models \neg\sigma^{WF}$. Then the wellfounded part of $\mathcal{M}$ is a model of ZFC and satisfies $\neg\sigma$. Hence ZFC $\nvDash \sigma$. $\square$

An Isomorphism...

Goal: Show that ${}^*A \cong A^{\mathbb{N}}/U_\alpha$.

Recall that we asked the question "Resemblances... Is agreeing on a "big set" enough?", and we had the following result.

Suppose $\varphi[\alpha] = \psi[\alpha]$ and let $\Lambda = \{n : \varphi(n) = \psi(n)\}$. Then for all sequences ζ and η we have

1. If $\zeta(n) = \eta(n)$ for all $n \in \Lambda$, then $\zeta[\alpha] = \eta[\alpha]$.
2. If $\zeta(n) \in \eta(n)$ for all $n \in \Lambda$, then $\zeta[\alpha] \in \eta[\alpha]$.

We'll see now that the answer to the above question is positive, and we'll see what we mean by "agreeing on a big set". Indeed, Λ is big if it's in the ultrafilter U_α.

Ace of Spades...

Theorem

The Alpha-Theory proves the following. For any nonempty set A, the map $K_A : \varphi(\alpha) \mapsto [\varphi]$ is an isomorphism from ${}^*A$ to $A^{\mathbb{N}}/U_\alpha$.

Proof: Let φ, ψ be sequences. First, it's claimed that

$$\varphi[\alpha] = \psi[\alpha] \iff \{n : \varphi(n) = \psi(n)\} \in U_\alpha \iff [\varphi] = [\psi].$$

For let $\Lambda = \{n : \varphi(n) = \psi(n)\}$. Clearly if $\{n : \varphi(n) = \psi(n)\} \in U_\alpha$ then $\varphi[\alpha] = \psi[\alpha]$. On the other hand, if $\varphi[\alpha] = \psi[\alpha]$, then since Λ is "big" and $\Lambda = \{n : 1_{\mathbb{N}}(n) \in \Lambda\}$, it follows that $\Lambda \in U_\alpha$. Also, the second biconditional is for free. This proves the claim. Next it's claimed that

$$\varphi[\alpha] \in \psi[\alpha] \iff \{n : \varphi(n) \in \psi(n)\} \in U_\alpha.$$

This time we put $\Lambda = \{n : \varphi(n) \in \psi(n)\}$ and repeat the above argument.$\square$

Cauchy's Infinitesimal Principles...

Definition

Cauchy's Infinitesimal Principle (CIP): Every infinitesimal number is the value at infinity of some infinitesimal sequence.

Definition

Strong Cauchy's Infinitesimal Principle (SCIP): Every nonzero infinitesimal number is the value at infinity of some monotone sequence.

The strength of the Alpha-Theory and Cauchy's Principles...

We are now at the heart of this exposition (what I wanted to study and understand). The aim is to understand the proofs of the following facts.

- Any theorem in "ordinary maths" is proved by the Alpha-Theory if and only if it it "true".
- By assuming the Alpha-Theory, we cannot prove nor disprove CIP.
- Assume Alpha-Theorey + CIP. Then we cannot prove nor disprove SCIP.
- Alpha-Theory + SCIP is consistent, if ZFC is consistent.

P-points and Selective Ultrafilters...

Definition

A non-principal ultrafilter D on $\mathbb{N}$ is a P-point (a selective ultrafilter) if every function on $\mathbb{N}$ becomes finite-to-one[a] or constant (1-1 or constant, respectively) if restricted to some suitable set in D.

[a] This means each preimage is finite.

Prof. Ross, N. Cutland, C. Kessler, and E. Kopp have shown that for the existence of an ultrapower where every infinitesimal is originated by some infinitesimal sequence requires a P-point.

A Characterization...

Let D be a non-principal ultrafilter on $\mathbb{N}$. Then,

- D is a P-point iff every infinitesimal in the ultrapower $\mathbb{R}^{\mathbb{N}}/D$ is the D-equivalence class of some infinitesimal sequence.
- D is selective iff every infinitesimal in the ultrapower $\mathbb{R}^{\mathbb{N}}/D$ is the D-equivalence class of some infinitesimal monotone sequence.

What if U_α is a P-point or a Selective Ultrafilter?

(CIP): Every infinitesimal number is the value at infinity of some infinitesimal sequence.

(SCIP): Every nonzero infinitesimal number is the value at infinity of some monotone sequence.

Theorem

The Alpha-Theory proves the following:

1. CIP *holds iff* U_α *is a* P*-point.*
2. SCIP *holds iff* U_α *is a selective ultrafilter.*

Proof: $K : \phi[\alpha] \mapsto [\phi]$ gives an isomorphism between ${}^*\mathbb{R}$ and $\mathbb{R}^{\mathbb{N}}/U_\alpha$. Now, by the previous characterization, CIP holds iff every infinitesimal in $\mathbb{R}^{\mathbb{N}}/U_\alpha$ is the U_α-equivalence class of some infinitesimal sequence iff U_α is a P-point. Proof of 2 is similar.

Ending it...

The paper ends with the next theorem. Apparently, it's based on the following facts about P-points and selective ultrafilters.

- There are ultrafilters which are not P-points.
- Selective ultrafilters exist if the continuum hypothesis is true.
- P-points which are not selective ultrafilters exist given the continuum hypothesis.
- There are models of ZFC with no P-points.

Ending it...

(CIP): Every infinitesimal number is the value at infinity of some infinitesimal sequence.

(SCIP): Every nonzero infinitesimal number is the value at infinity of some monotone sequence.

Theorem

1. *Alpha-Theory does not prove* CIP*.*
2. *Alpha-Theory* + CIP *does not prove* SCIP*.*
3. *Alpha-Theory* + SCIP *is consistent, if* ZFC *is consistent.*

Part 1 of the theorem is really a surprise (to me) because we can imagine infinitesimals using infinitesimal sequences, but according to the theorem there might be infinitesimals that we can't imagine to be ideal values of infinitesimal sequences!

Ending it...

To do: Think (more) about the following.

1. Łoś's theorem and a "suitable ultrapower" in the proof of transfer. Is the ultrapower $V^{\mathbb{N}}/U_\alpha$, where V is the universe of all mathematical objects in α-theory?
2. The actual proof of the theorem about ZFBC.
3. Does an internal definition principle for α-theory make sense? Can we use the isomorphism ${}^*A \cong A^{\mathbb{N}}/U_\alpha$ to get one?
4. P-points and selective ultrafilters.